Mutation semigroup algebras:

We work over a field $\mathbb{K}$. Let $\mathbb{F} \supseteq \mathbb{K}$ be a transcendental extension of degree n.

Definition: A "seed" $\{(x_1, \dots, x_n), B\}$ is a set of n elements in $\mathbb{F}$ algebraically independent over $\mathbb{K}$, called "cluster variables" and $B = \{b_{ij}\}$ a $n \times n$ skew-symmetric matrix with integral entries called the "exchange matrix".

Definition: A "cluster mutation" is a way to produce a new seed from a starting seed $\{(x_1, \dots, x_n), B\}$ and the choice of $j \in \{1, \dots, n\}$. We delete x_j from the cluster variables and add

$$x_j' := x_j^{-1} \left(\prod_{b_{ij}>0} x_i^{b_{ij}} + \prod_{b_{ij}<0} x_i^{-b_{ij}} \right)$$

Then, we replace B with B' defined by

$$b'_{ik} := \begin{cases} b_{ki} & \text{if } k=j \\ b_{ik} - b_{ij} b_{jk} & \text{if } k \neq j,\ b_{ij}<0,\ b_{jk}<0 \\ b_{ik} + b_{ij} b_{jk} & \text{if } k \neq j,\ b_{ij}>0,\ b_{jk}>0. \\ b_{ik} & \text{otherwise.} \end{cases}$$

The exchange $\{(x_1, \dots, x_n), B\} \longmapsto \{(x_1, \dots, x_j', \dots, x_n), B'\}$ is the "cluster mutation"

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Definition: A "cluster algebra" $R \subseteq \mathbb{F}$ is the $\mathbb{K}$-algebra generated by all the cluster variables of all seeds that can be obtained by cluster mutations from an initial seed

Example: Let $\mathbb{F} = \mathbb{K}(x_1, x_2)$, and consider the initial seed $\left\{ (x_1, x_2), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$. Then, we can compute

$$x_3 := x_1' = \frac{1 + x_2}{x_1}, \quad B' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$x_4 := x_2' = \frac{1 + x_3}{x_2} = \frac{1 + \left(\frac{1 + x_2}{x_1}\right)}{x_2} = \frac{1 + x_1 + x_2}{x_1 x_2}$$

$$x_5 := x_3' = \frac{1 + x_4}{x_3} = \frac{1 + \left(\frac{1 + x_1 + x_2}{x_1 x_2}\right)}{\left(\frac{1 + x_2}{x_1}\right)} = \frac{x_1 + 1}{x_2}$$

We also get. $x_6 := x_4' = x_1$, $x_7 = x_5' = x_2$, ... cyclically.

Hence, we conclude

$$R = \mathbb{K}\left[x_1, x_2, \frac{1 + x_2}{x_1}, \frac{1 + x_1 + x_2}{x_1 x_2}, \frac{1 + x_1}{x_2} \right] \subseteq \mathbb{K}[x_1^{\pm}, x_2^{\pm}].$$

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Theorem (Fomin-Zelevinsky, Laurent Phenomenon):

Let R be a cluster algebra. Let $\{(x_1, \dots, x_n), B\}$ be any seed of R. Then $R \subseteq \mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]$.

Geometrically: Assume for simplicity R is f.g over $\mathbb{K}$.

Each inclusion of the LP induce an embedding $i_j : \mathbb{G}_m^n \hookrightarrow X$, where $X = \mathrm{Spec}(R)$. Furthermore, if we let

$$\Omega_n := \frac{dx_1 \wedge \dots \wedge dx_n}{x_1 \cdots x_n}$$

be the so-called volume form of the algebraic torus, then

$(i_1 \circ i_2^{-1})^* \Omega_n = \Omega_n$ holds for any two such torus embeddings i_1 & i_2.

Furthermore, $\bigcup_{\substack{\text{seeds} \\ \text{of } R}} i_j(\mathbb{G}_m^n) \subseteq X$ is a "big" open subset, i.e., its complement is a closed subset of codimension ≥ 2.

Remark: Geometrically cluster varieties, i.e., spectrum of cluster algebras, can be understood as union of algebraic tori such that the gluing functions respect the volume form.

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Definition: A toric variety is a n-dimensional variety X that admits an open embedding $\mathbb{G}_m^n \hookrightarrow X$ such that the action of $\mathbb{G}_m^n$ on itself extends to an action on X.

Theorem (Enwright - Figueroa - M, 2024): Let X be a normal variety. Assume that X is either affine or projective. Then X is toric if and only if there exists and embedding $\mathbb{G}_m^n \xhookrightarrow{i} X$ such that Ω_n has poles along every prime component of $X \setminus i(\mathbb{G}_m^n)$.

Remark: By the previous theorem, both; cluster varieties and toric varieties can be understood via the volume form of the torus.

Aim: Define the "smallest" class of $\mathbb{K}$-algebras which contains both cluster algebras and semigroup algebras.

From now on, M will denote a finitely generated free abelian group, N will denote its dual and $M_{\mathbb{Q}}$, $N_{\mathbb{Q}}$ be the corresponding $\mathbb{Q}$-vector spaces.

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Definition: An "algebraic mutation datum" is a pair $(u, h = g^{k})$, $k \geqslant 0$, where $u \in N$ is primitive and $g \in \mathbb{K}[u^{\perp} \cap M]$. We say that the algebraic mutation datum is "σ-admissible" if $u \notin \sigma$.

Definition: Let $\mu : U_{\sigma_1} \dashrightarrow U_{\sigma_2}$ be a birational map between affine normal toric varieties corresponding to the cones $\sigma_1, \sigma_2 \subseteq M_{\mathbb{Q}}$. We say that μ is an "algebraic mutation" if:

1) we have $\mu^{*} : \mathbb{K}(M) \xrightarrow{\sim} \mathbb{K}(M)$ is given by

$\mu^{*}(x^{m}) = x^{m} h^{-\langle u, m \rangle}$ for some σ_2 - admissible (u, h), &

2) μ induces a bijection between torus invariant divisors.

Definition: A "mutation semigroup algebra" is a f.g. ring R over $\mathbb{K}$. that can be written as $R = R_0 \cap \ldots \cap R_n$ such that the following holds for any $i \in \{0, \ldots, n\}$:

1) there is $j_i : \mathbb{K}[\sigma_i^{\vee} \cap M] \hookrightarrow \mathrm{Frac}(R)$ inducing $\mathbb{K}(M) \cong \mathrm{Frac}(R)$ and with image R_i;

2) $j_0 \circ j_i^{-1}$ is an algebraic mutation; and

3) if $p \subseteq R_i$ is a prime of height one, then so does $R \cap p$ in R.

We abbreviate mutation semigroup algebras by MSA. If $\sigma_i = \{0\}$ for each i, then we just call it a mutation algebra

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Example: $\mathbb{K}[x,y] \cap \mathbb{K}[\frac{y+1}{x}, y] \cong \mathbb{K}[x,y,z]/(xz-y-1)$

is an example of a MSA.

Remark: MSA are rather general compared to cluster algebras. Because of the existence of the cones σ_i, their spectrum may exhibit some toric singularities. Moreover, the mutations are much more general and may be non-irreducible polynomials. Finally, the definition is "seed-free" so the Laurent Phenomenon is not well defined & in some contexts may not hold

Some problems of interest:

(A) Are MSA strongly F-regular? F-pure?

(B) Do MSA satisfy the geometric vertex decomposition?

(C) How to describe symbolic powers of ideals on MSA?

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MSA are deeply connected to Fano geometry.
The connection is stablished via the following theorems.

Theorem (Enwright - Francone - M - Spink, 2025): Let R be a finitely generated MSA. Assume $U = \mathrm{Spec}(R)$ has klt sings. Then, there is a klt Fano compactification $U \hookrightarrow X$.

Thus, every MSA "appears" inside some Fano variety.
Conversely, many Fano varieties are compactifications of spectrum of MSA. More precisely, we have:

Theorem (Gross - Hacking - Keel 2006 + Avilov - Loginov - Przyjalkowski 2023):
A general smooth Fano surface is a compactification of $\mathrm{Spec}(\mathrm{MSA})$.

The following is work in progress, and the actual number may increase by the time of publication.

Theorem (Alves da Silva - Figueroa - M, 26):
There are ≥ 70 families of smooth Fano threefolds whose general elements is a compactification of $\mathrm{Spec}(\mathrm{MSA})$

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Finally, we discuss a projective version of MSA, the so-called "cluster type varieties".

Definition: We say that a map $i: U \dashrightarrow X$ is an "embedding in codimension one" if there exists $Z \subsetneq U$ closed of codimension ≥ 2 such that $i|_{U \setminus Z}$ is an embedding.

Definition: A normal variety X is said to be "cluster type" if there exists an embedding in codimension one $\mathbb{G}_m^n \overset{i}{\dashrightarrow} X$ such that Ω_n has no zeros on X. Let $B = \text{poles}(\Omega_n)$. Then, we call $B \geq 0$ the "cluster type boundary" and (X, B) a "cluster type pair".

The following theorem states that the geometric behavior previously observed for spectra of cluster algebras also holds for cluster type var.

Theorem (Cort: 2022, EFM 2024): Let (X, B) be a cluster type pair, then $X \setminus B$ is covered, up to a closed subset of codimension ≥ 2, by images of embeddings in codimension one $j_i : \mathbb{G}_m^n \dashrightarrow X$. Furthermore, for any two such maps j_1 & j_2, we have $(j_1 \circ j_2^{-1})^* \Omega_n = \Omega_n$.

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MSA essentially give natural charts for cluster type varieties.

Theorem (EFMS25): Let R be a f.g ring over $\mathbb{K}$. Assume that $U = \mathrm{Spec}(R)$ has klt singularities. Then, the two following conditions are equivalent:

1) R is a MSA; &

2) there exists a cluster type pair (X, B) & $A \leq B$ ample such that $R \cong H^0(X/A, \mathcal{O}_{X/A})$.

Remark: Spectra of MSA are to cluster type varieties what affine toric varieties are to toric varieties.